Optimal quantum preparation contextuality in *n*-bit parity-oblivious multiplexing task

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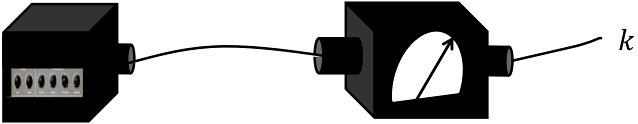
Outline of the talk

1. Ontological model of an operational theory
2. Parity oblivious multiplexing (POM) task in brief
3. 3-bit quantum POM task
4. *n*-bit quantum POM task
5. Summary

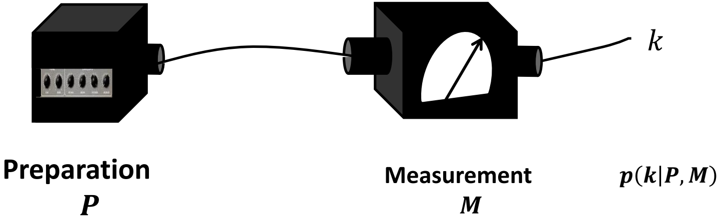
Outline of the talk

Operational theory



$p(k|P, M) \Rightarrow$ Probability of occurance of outcome $k \in \mathcal{K}_M$ with $P \in \mathcal{P}$ and $M \in \mathcal{M}$.

Ontological model of operational theory



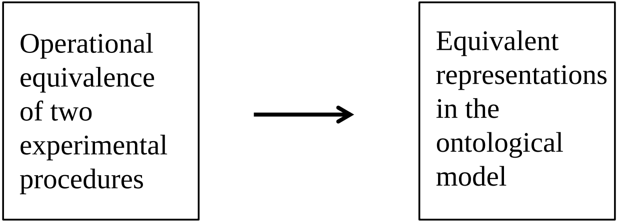
Preparation procedure P prepares ontic state $\lambda \in \Lambda$ according to the distribution $\mu(\lambda|P)$ satisfying $\int_\Lambda \mu(\lambda|P) d\lambda = 1$.

Given a M, λ assings a response function $\xi(k|M,\lambda)$ satisfying $\sum_{k \in \mathcal{K}_\mathcal{M}} \xi(k|M,\lambda) = 1$.

$$p(k|P,M) = \int_{\lambda \in \Lambda} \mu(\lambda|P)\xi(k|M,\lambda) d\lambda$$

Equivalence class of experimental procedures

An ontological model of operational theory is non-contextual if



Spekkens, PRA (2005).

Equivalence class of of experimental procedures

Equivalence class of Measurements $(M \equiv M')$**:**

$$\forall k, \forall P \;\; p(k|P, M) = p(k|P, M') \quad \Rightarrow \quad \xi(k|M, \lambda) = \xi(k|M', \lambda)$$

Equivalence class of Preparations $(P \equiv P')$**:**

$$\forall k, \forall M \;\; p(k|P, M) = p(k|P', M) \quad \Rightarrow \quad \mu(\lambda|P) = \mu(\lambda|P')$$

Assumption of preparation non-contextuality (PNC)in ontological model.

Ontological model of Quantum Theory

Preparation produces ρ and measurement is described by POVM E_k.

$$p(k|P, M) = Tr[\rho E_k] \text{ (Born-Dirac rule)}$$

Ontological model of QM:

$$P \leftrightarrow \mu_P(\lambda|\rho)$$

$$M \leftrightarrow \xi_M(k|\lambda, E_k).$$

$$\forall \rho, \forall E_k, \forall k \qquad \int_\Lambda \mu_P(\lambda|\rho)\xi_M(k|\lambda, E_k)d\lambda = Tr[\rho E_k]$$

Ontological model of QM and non-contextuality

- **Measurement non-contextual**

$$\forall P, \forall k \quad p(k|P, M) = p(k|P, M') \quad \Rightarrow \quad \xi_M(k|\lambda, E_k) = \xi_{M'}(k|\lambda, E_k)$$

M and M' are two disticnt precedures realizing E_k.

Kochen-Speaker Non-contextuality: Measurement non-contextuality + outcome determinism for sharp measurement

Measurement non-contextuality (along with or without determinism) and its quantum violation has been extensively studied.

Ontological model of QM and non-contextuality

- **Preparation non-contextual**

$\forall M, \forall k \qquad p(k|P,M) = p(k|P',M) \quad \Rightarrow \quad \mu_P(\lambda|\rho) = \mu_{P'}(\lambda|\rho)$

P and P' are two distinct preparation procedures.

Ex:

$$\begin{aligned} \rho = \mathbb{I}/2 &= (|0\rangle\langle 0| + |0\rangle\langle 0|)/2 \\ &= (|+\rangle\langle +| + |-\rangle\langle -|)/2 \end{aligned} \tag{1}$$

$$\mu_P(\lambda|\mathbb{I}/2) = \mu_{P'}(\lambda|\mathbb{I}/2)$$

Outline of the talk

Parity oblivious multiplexing (POM) task

It is variant of Random Access Code.

1. Alice has an n-bit string x chosen uniformly at random from $\{0, 1\}^n$.
2. Bob chooses any bit $y \in \{1, 2, ..., n\}$ and recover the bit x_y with a probability.
3. The condition of the task is, Bob's output must be the bit $b = x_y$ (y^{th} bit of Alice's input string).
4. Alice and Bob try to optimize the guessing probability $P(b = x_y)$.
5. Parity oblivious constraint: no information about any parity of x can be transmitted to Bob.

POM task

Define a parity set $\mathbb{P}_n := \{z|z \in \{0,1\}^n, \sum_y z_y \geq 2\}$.

Parity oblivious constraint: For any $s \in \mathbb{P}_n$, no information about $s.x = \oplus_y s_y x_y$ (s-parity) is to be transmitted to Bob, where $\oplus$ is sum modulo 2.

We then have a s-parity 0 set and a s-parity 1 set.

Ex: For $\{0,1\}^2$, $\mathbb{P} = \{11\}$. No information about $s.x = x_1 \oplus x_2$ will be transmitted to Bob.

$s-$parity 0 set is $\{00, 11\}$ and $s-$parity 1 set is $\{01, 10\}$

R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner and G. J. Pryde Phys. Rev. Lett. 102, 010401 (2009).

Average success probability of POM task

The average success probability is

$$p(b = x_y) = \frac{1}{2^n n} \sum_{x,y} p(b = x_y | P_x, M_y). \tag{2}$$

The parity-obliviousness condition guarantees that there are no outcome of any measurement for which the probabilities for s-parity 0 and s-parity 1 are different. Mathematically,

$$\forall s \, \forall M \, \forall k \sum_{x|x.s=0} p(P_x|k, M) = \sum_{x|x.s=1} p(P_x|k, M). \tag{3}$$

It is proved that

$$p(b = x_y) \leq \frac{n+1}{2n} \tag{4}$$

Average success probability of POM task

Let Alice always tells Bob the first bit but still no information about the parity is transmitted.

- If $y = 1$, occurring with probability $1/n$, Bob can predict the outcome with certainty
- If $y \neq 1$, occurring with probability of $\frac{n-1}{n}$, he at best guesses the bit with probability $1/2$.
- The total probability of success is $1/n + (n-1)/2n = (n+1)/2n$.

PNC model and POM task

The parity oblivious condition in operational theory is PNC assumption in ontological model.

In a preparation non-contextual (PNC) model,

$$\sum_{x|x.s=0} p(\lambda|P_x) = \sum_{x|x.s=1} p(\lambda|P_x)$$

The success probability in a PNC model for n-bit POM task satisfies

$$p(b = x_y)_{pnc} \leq \frac{n+1}{2n}$$

R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner and G. J. Pryde Phys. Rev. Lett. 102, 010401 (2009).

Quantum POM task

In QM, Alice encodes her n-bit string of $x \in \{1, 2, ..., 2^n\}$ into pure quantum states ρ_x, prepared by procedure P_x.

After receiving the state ρ_x, for every $y \in \{1, 2, ..., n\}$, Bob performs a dichotomic measurement M_y and reports the outcome b as his output.

- For 2-bit POM task: $p_Q^{opt} = (1/2)(1 + 1/\sqrt{2}) > p(b = x_y)_{pnc} = 3/4$.
- 3-bit POM task: $p_Q = (1/2)(1 + 1/\sqrt{3})$ but left its optimality as open question.

R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner and G. J. Pryde, Phys. Rev. Lett. 102, 010401 (2009).

Quantum POM task

- Even POM task is shown to be equivalent to an INDEX game if some conditions are additionally satisfied and then using semidefinite program p_Q is optimized. Optimal value is $p_Q^{opt} = (1/2)(1 + 1/\sqrt{n})$.

A. Chailloux, I. Kerenidis, S. Kundu and J. Sikora, New J. Phys.18, 045003(2016).

- $2-$bit POM task reduces to a CHSH game and p_Q^{opt} is obtained at optimal violation of CHSH inequality.

M. Banik *et al.*, Phys. Rev. A, 92, 030103(R) (2015).

- n-bit POM with d outcomes has been studied and showed $P(b = x_y)_{pnc} \leq (n + d - 1)/nd$. As regards the quantum POM task, they provide explicit example for $d = 3$ which reduces to CGLMP game.

A. Hameedi, A. Tavakoli, B. Marques and M. Bourennane, PRL, 119, 220402 (2017)

Quantum POM task

- We adopt similar approach of Banik *et al.* to obtain the optimal success probability for $n-$bit POM task.
- We first demonstrate that 3-bit quantum POM task reduces to task of optimaizing the quantum vioation of (4×3) elegant Bell's inequality [Gisin].
- For $n-$bit quantum POM task, the success probability can be shown to be dependent on a curious form of $(2^{n-1} \times n)$ elegant Bell's inequality .
- We use an interesting approach to optimize the quantum violation of $(2^{n-1} \times n)$ inequality.

N.Gisin, arXiv:quant-ph/0702021

Outline of the talk

3- bit quantum POM task

Parity set $\mathbb{P}_n = \{z | z \in \{0,1\}^n, \sum_y z_y \geq 2\}$.

PO condition: $s \in \{011, 101, 110, 111\}$, no information about $s.x = \oplus_y s_y x_y$ (s-parity) will be transmitted to Bob.

For s=110, the $s-$parity 0 set is $\{000, 001, 110, 111\}$ and $s-$parity 1 set is $\{010, 100, 011, 101\}$.

The parity obliviousness in QM:

$$\begin{aligned} \rho &= \frac{1}{4}(\rho_{000} + \rho_{111} + \rho_{110} + \rho_{001}) \\ &= \frac{1}{4}(\rho_{010} + \rho_{101} + \rho_{011} + \rho_{100}) \end{aligned} \tag{5}$$

3- bit quantum POM task

Consider a shared entangled state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| \in \mathbb{C}^d \otimes \mathbb{C}^d$.

Alice's projective measurements: $\{P_{A_i}, I - P_{A_i}\}$ where $i = 1, 2, 3, 4$

$$
\begin{aligned}
\frac{1}{2}\rho_{000} &= Tr_1\Big[\left(P_{A_1} \otimes I\right) \rho_{AB}\Big] & \frac{1}{2}\rho_{111} &= Tr_1\Big[\left(I - P_{A_1} \otimes I\right) \rho_{AB}\Big]\\
\frac{1}{2}\rho_{001} &= Tr_1\Big[\left(P_{A_2} \otimes I\right) \rho_{AB}\Big] & \frac{1}{2}\rho_{110} &= Tr_1\Big[\left(I - P_{A_2} \otimes I\right) \rho_{AB}\Big]\\
\frac{1}{2}\rho_{010} &= Tr_1\Big[\left(P_{A_3} \otimes I\right) \rho_{AB}\Big] & \frac{1}{2}\rho_{101} &= Tr_1\Big[\left(I - P_{A_3} \otimes I\right) \rho_{AB}\Big]\\
\frac{1}{2}\rho_{100} &= Tr_1\Big[\left(P_{A_4} \otimes I\right) \rho_{AB}\Big] & \frac{1}{2}\rho_{011} &= Tr_1\Big[\left(I - P_{A_4} \otimes I\right) \rho_{AB}\Big].
\end{aligned}
$$

3- bit quantum POM task

After receiving the particle from Alice, Bob performs three projective measurements $\{P_{B_y}, I - P_{B_y}\}$ in order to guess the Alice's bit, with $y = 1, 2, 3$.

If qubit system used for encoding and decoding, $p_Q = (1/2)\big(1 + 1/\sqrt{3}\big)$

$$A_1 = (\sigma_x + \sigma_y + \sigma_z)/\sqrt{3}, \qquad A_2 = (\sigma_x + \sigma_y - \sigma_z)/\sqrt{3}$$

$$A_3 = (\sigma_x - \sigma_y + \sigma_z)/\sqrt{3}, \qquad A_4 = (-\sigma_x + \sigma_y + \sigma_z)/\sqrt{3}$$

$$B_1 = \sigma_x, \quad B_2 = -\sigma_y \text{ and } B_3 = \sigma_z$$

Is P_Q optimal ?

R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner and G. J. Pryde, Phys. Rev. Lett. 102, 010401 (2009).

3- bit quantum POM task

Let Alice and Bob share the entangled state $|\Psi_{AB}\rangle$. Then,

$$
\begin{aligned}
p_Q = \frac{1}{24}\Big(& Tr[\rho_{000}P_{B_1}] + Tr[\rho_{000}P_{B_2}] + Tr[\rho_{000}P_{B_3}] \\
& + Tr[\rho_{001}P_{B_1}] + Tr[\rho_{001}P_{B_2}] + Tr[\rho_{001}(I - P_{B_3})] \\
& + Tr[\rho_{010}P_{B_1}] + Tr[\rho_{010}(I - P_{B_2})] + Tr[\rho_{010}P_{B_3}] \\
& + Tr[\rho_{100}(I - P_{B_1})] + Tr[\rho_{100}P_{B_2}] + Tr[\rho_{100}P_{B_3}] \\
& + Tr[\rho_{011}P_{B_1}] + Tr[\rho_{011}(I - P_{B_2})] + Tr[\rho_{011}(I - P_{B_3})] \\
& + Tr[\rho_{101}(I - P_{B_1})] + Tr[\rho_{101}P_{B_2}] + Tr[\rho_{101}(I - P_{B_3})] \\
& + Tr[\rho_{110}(I - P_{B_1})] + Tr[\rho_{110}(I - P_{B_2})] + Tr[\rho_{110}P_{B_3}] \\
& + Tr[\rho_{111}(I - P_{B_1})] + Tr[\rho_{111}(I - P_{B_2})] + Tr[\rho_{111}(I - P_{B3})]\Big) \qquad (6)
\end{aligned}
$$

3- bit quantum POM task

Quantum success probability:$p_Q = \frac{1}{2}\left(1 + \frac{\langle \mathcal{B}_3 \rangle_Q}{12}\right)$

where

$$\begin{aligned}\mathcal{B}_3 =& (A_1 + A_2 + A_3 - A_4) \otimes B_1 \\ &+ (A_1 + A_2 - A_3 + A_4) \otimes B_2 \\ &+ (A_1 - A_2 + A_3 + A_4) \otimes B_3\end{aligned}$$

In a local model, $\mathcal{B}_3 \leq 6 \quad \Rightarrow \quad$ The elegant Bell inequality[Gisin].

N.Gisin, arXiv:quant-ph/0702021

3-bit quantum POM task

Optimal value of $\mathcal{B}_3$ provides the optimal p_Q.

Define $\gamma_3 = 4\sqrt{3}\,\mathbb{I} - \mathcal{B}_3$ such that $\gamma_3 = (\sqrt{3}/2)\sum_{k=1}^{4} M_k^\dagger M_k$

$$M_1 = (B_1 + B_2 + B_3)/\sqrt{3} - A_1, \quad M_2 = (B_1 + B_2 - B_3)/\sqrt{3} - A_2$$
$$M_3 = (B_1 - B_2 + B_3)/\sqrt{3} - A_3, \quad M_4 = (-B_1 + B_2 + B_3)/\sqrt{3} - A_4$$

Since γ_3 is positive semidefinite, then $\mathcal{B}_3 = 4\sqrt{3}\,\mathbb{I} - \gamma_3 \leq 4\sqrt{3}\mathbb{I}$,

$$p_Q \leq (1/2)(1 + (1/\sqrt{3}))$$

A. Acin, S. Pironio, T. Vertesi, and P. Wittek, Phys. Rev. A 93, 040102(R) (2016)

3- bit quantum POM task

Quantum success probability: $$p_Q = \frac{1}{2}\left(1 + \frac{\langle \mathcal{B}_3 \rangle_Q}{12}\right)$$

$$\begin{aligned}\mathcal{B}_3 =& (A_1 + A_2 + A_3 - A_4) \otimes B_1 \\ &+ (A_1 + A_2 - A_3 + A_4) \otimes B_2 \\ &+ (A_1 - A_2 + A_3 + A_4) \otimes B_3\end{aligned}$$

One can write

$$\begin{aligned}(A_1 + A_2 + A_3 - A_4) \otimes \mathbb{I} &= \sqrt{3} B_1 \otimes \mathbb{I} \\ (A_1 + A_2 - A_3 + A_4) \otimes \mathbb{I} &= \sqrt{3} B_2 \otimes \mathbb{I} \\ (A_1 - A_2 + A_3 + A_4) \otimes \mathbb{I} &= \sqrt{3} B_3 \otimes \mathbb{I}\end{aligned}$$

Then, $\mathcal{B}_3 = \frac{4}{\sqrt{3}}(B_1 \otimes B_1 + B_2 \otimes B_2 + B_3 \otimes B_3))$

Outline of the talk

n-bit quantum POM task

Alice chooses her bit x^δ randomly from $\{0,1\}^n$ with $\delta \in \{1, 2...2^n\}$.

The relevant set $\mathcal{D}_n = \{x^\delta | x^i \oplus x^j = 11...1, i + j = 2^n + 1\}$ where $i, j \in \delta$.
Here, $x^1 = 00...00, x^2 = 00...01, ...$

The parity set: $\mathbb{P}_n = \{x^\delta | x^\delta \in \{0,1\}^n, \sum_r x_r^\delta \geq 2\}$ and we consider $x^s = 1100...00$.

Aice performs 2^{n-1} projective measurements $\{P_{A_i}, \mathbb{I} - P_{A_i}\}$ on the entangled state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ to encode her n-bits into 2^n pure quantum states are

$$\rho_{x^i} = tr_A[(P_{A_i} \otimes \mathbb{I})\rho_{AB}]; \qquad \rho_{x^j} = tr_A[(\mathbb{I} - P_{A_i} \otimes \mathbb{I})\rho_{AB}] \tag{7}$$

n-bit quantum POM task

Bob's measurements:

$$M_y^{i,j} = \begin{cases} P_{B_y}, & \text{when } x_y^{i,j} = 0 \\ \mathbb{I} - P_{B_y}, & \text{when } x_y^{i,j} = 1 \end{cases}$$

The success probability

$$p_Q = \frac{1}{2^n n} \sum_{y=1}^{n} \sum_{i=1}^{2^{n-1}} p(b = x_y^i | \rho_{x^i}, M_y^i) + p(b = x_y^j | \rho_{x^j}, M_y^j)$$

n-bit quantum POM task

In QM, we have

$$p_Q = \frac{1}{2^n n}\sum_{y=1}^{n}\sum_{i=1}^{2^{n-1}} tr[\rho_{x^i} M_y^i] + tr[\rho_{x^j} M_y^j]$$

Since $\forall i, j$, $x^i \oplus x^j = 111...111$ we have $x_y^i \oplus x_y^j = 1$, then

$$p_Q = \frac{1}{2^n n}\sum_{y=1}^{n}\sum_{i=1}^{2^{n-1}} (-1)^{x_y^i} tr[(\rho_{x^i} - \rho_{x^j})P_{B_y}] + tr[\rho_{x^{(i.x_y^i + j.x_y^j)}}]$$

$$= \frac{1}{2} + \frac{1}{2^n n}\sum_{i,y}(-1)^{x_y^i}\langle A_i \otimes B_y\rangle + \frac{1}{2^n n}\sum_{i,y}(-1)^{x_y^i}\langle A_i \otimes I\rangle$$

n-bit quantum POM task

Optimaizing $\langle \mathcal{B}_n \rangle_Q = \sum_{i,y} (-1)^{x_y^i} \langle A_i \otimes B_j \rangle$ optimizes the sucess probability.

$$\text{In a local model,} \qquad \mathcal{B}_n \leq \sum_{r=0}^{[\frac{n-1}{2}]} (n-2r)(nr)$$

In order to optimize $\mathcal{B}_n$ we define,

$$\gamma_n = 2^{n-1} \sqrt{n}\, \mathbb{I} - \mathcal{B}_n$$

γ_n can be written as

$$\gamma_n = \frac{\sqrt{n}}{2} \sum_{k=1}^{2^{n-1}} M_k^\dagger M_k; \text{ with } M_k = \sum_y (-1)^{x_y^k} \frac{B_y}{\sqrt{n}} - A_k$$

n-bit quantum POM task

If $A_k^\dagger A_k = \mathbb{I} = B_y^\dagger B_y$, then γ_n is positive semidefinite.

Which in turn provides

$$\mathcal{B}_n \leq 2^{n-1}\sqrt{n}\,\mathbb{I}$$

Hence, the quantum success probability for n-bit POM task satisfies

$$p_Q \leq \frac{1}{2}\left(1 + \frac{1}{\sqrt{n}}\right).$$

Can $n-$bit POM task be optimized for qubit system?

When $n = 4$

$$p_Q = \frac{1}{2}\left(1 + \frac{\langle \mathcal{B}_4 \rangle}{32}\right)$$

$$\begin{aligned}
\mathcal{B}_4 &= (A_1 + A_2 + A_3 + A_4 - A_5 + A_6 + A_7 + A_8) \otimes B_1 \\
&+ (A_1 + A_2 + A_3 - A_4 + A_5 + A_6 - A_7 - A_8) \otimes B_2 \\
&+ (A_1 + A_2 - A_3 + A_4 + A_5 - A_6 + A_7 - A_8) \otimes B_3 \\
&+ (A_1 - A_2 + A_3 + A_4 - A_5 - A_6 - A_7 + A_8) \otimes B_4
\end{aligned}$$

$$\mathcal{B}_4 = \frac{8}{\sqrt{4}} \sum_{i=1}^{4} B_i \otimes B_i$$

When $n = 4$

$B_1 = \sigma_x \otimes \sigma_x, B_2 = -\sigma_x \otimes \sigma_y, B_3 = \sigma_x \otimes \sigma_z$ and $B_4 = -\sigma_y \otimes I$.

$$A_1 = \frac{1}{2}(\sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_y + \sigma_x \otimes \sigma_z + \sigma_y \otimes I)$$

$$A_2 = \frac{1}{2}(\sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_y + \sigma_x \otimes \sigma_z - \sigma_y \otimes I)$$

$$A_3 = \frac{1}{2}(\sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_y - \sigma_x \otimes \sigma_z + \sigma_y \otimes I)$$

$$A_4 = \frac{1}{2}(\sigma_x \otimes \sigma_x - \sigma_x \otimes \sigma_y + \sigma_x \otimes \sigma_z + \sigma_y \otimes I)$$

$$A_5 = \frac{1}{2}(-\sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_y + \sigma_x \otimes \sigma_z + \sigma_y \otimes I)$$

$$A_6 = \frac{1}{2}(\sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_y - \sigma_x \otimes \sigma_z - \sigma_y \otimes I)$$

$$A_7 = \frac{1}{2}(\sigma_x \otimes \sigma_x - \sigma_x \otimes \sigma_y + \sigma_x \otimes \sigma_z - \sigma_y \otimes I)$$

$$A_8 = \frac{1}{2}(\sigma_x \otimes \sigma_x - \sigma_x \otimes \sigma_y - \sigma_x \otimes \sigma_z + \sigma_y \otimes I)$$

Outline of the talk

Summary

- We studied the n-bit quantum POM task. The quantum success probability is shown to be larger than the PNC bound.
- We drive the optimal success probability for n-bit POM task by using an interesting approach
- For the POM task $n > 3$ qubit system is not useful. Higher dimensional system is requited to achieve the optimal bound.
- Our optimization approach may be generalised for d outcome scenrio which will be studied in future.

Thank you.